The Laplacian Spread of Tricyclic Graphs

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Basic definitions

• adjaceny matrix:

\[ A = A(G) = [a_{ij}]_{n \times n} = \begin{cases} 
1, & \text{if } v_i \text{ is adjacent to } v_j, \\
0, & \text{otherwise}. 
\end{cases} \]

• The spectrum of \( G \) can be denoted by \( S(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)) \), where \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \).

• The spread of the graph \( G \) is defined as

\[ \varphi_A(G) = \lambda_1(G) - \lambda_n(G). \]
Known results on the spread of graphs

  Petrović determines all minimal graphs whose spread do not exceed 4.

  Gregory, Hershkowitz and Kirkland present some lower and upper bounds for the spread of a graph. They show that the path is the unique graph with minimum spread among connected graphs of given order.

Li, Zhang and Zhou determine the unique graph with maximum spread among all unicyclic graphs with given order not less than 18, which is obtained from a star by adding an edge between two pendant vertices.

• Bolian Liu and Muhuo Liu, On the spread of the spectrum of a graph, Discrete Math. 309 (2009)

Bolian Liu and Muhuo Liu obtain some new lower and upper bounds for the spread of a graph, which are some improvements of Gregory’s bound on the spread for graphs with additional restrictions.
Basic definitions

• The Laplacian matrix of the graph $G$ is $L(G) = D(G) - A(G)$, where 
  $D(G) = \text{diag}(d(v_1), d(v_2), ..., d(v_n))$ denotes the diagonal matrix of vertex 
  degrees of $G$, and $d(v)$ denotes the degree of the vertex $v$ of $G$. 

• The Laplacian spectrum of $G$ can be denoted by 
  $$SL(G) = (\mu_1(G), \mu_2(G), ..., \mu_{n-1}(G)), \mu_n(G) = 0),$$

  where $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ are the eigenvalues of $L(G)$ arranged in 
  weakly decreasing order. 

• The Laplacian spread of the graph $G$ is defined as 
  $$\varphi_L(G') = \mu_1(G) - \mu_{n-1}(G').$$
Known results on the Laplacian spread of graphs

- **Y. Fan, J. Xu, Y. Wang, D. Liang, The Laplacian spread of a tree, Discrete Mathematics and Theoretical Computer Science, 10(1)(2008).**

  Yizheng Fan et al. have shown that among all trees of fixed order, the star is the unique one with maximum Laplacian spread and the path is the unique one with the minimum Laplacian spread.

- **Y. Bao, Y. Tan, Y. Fan, The Laplacian spread of unicyclic graphs, Applied Mathematics Letters, 22(2009).**

  Yizheng Fan et al. have shown that among all unicyclic graphs of fixed order, the unique unicyclic graph with maximum Laplacian spread is obtained from a star by adding an edge between two pendant vertices.

Yizheng Fan et al. have shown that among all bicyclic graphs of fixed order, the only two bicyclic graphs with maximum Laplacian spread are obtained from a star by adding two incident edges and by adding two nonincident edges between the pendant vertices of the star.

Main results: In this paper, we investigate Laplacian spread of graphs, and prove that there exist exactly five types of tricyclic graphs with maximum Laplacian spread among all tricyclic graphs of fixed order.

Some lemmas

- **Lemma 2.1** [1] Let $G$ be a connected graph of order $n \geq 2$. Then

$$\mu_1(G) \leq n,$$

with equality if and only if the complement graph of $G$ is disconnected.

- **Lemma 2.2** [3] Let $G$ be a connected graph with vertex set $\{v_1, v_2, ..., v_n\} (n \geq 2)$. Then

$$\mu_1(G) \leq \max\{d(v_i) + d(v_j) - |N(v_i) \cap N(v_j)| : v_iv_j \in E(G)\}.$$
• **Lemma 2.3** [12] Let $G$ be a connected graph with vertex set $\{v_1, v_2, \ldots, v_n\}(n \geq 2)$. Then

$$
\mu_1(G) \leq \max\{d(v_i) + m(v_i) : v_i \in V(G')\},
$$

where $m(v_i) = \frac{\sum_{v_j \in N(v_i)} d(v_j)}{d(v_i)}$, the average of the degrees of the vertices adjacent to $v_i$.

• **Lemma 2.4** [7] Let $G$ be a graph of order $n \geq 2$ containing at least one edge. Then

$$
\mu_1(G') \geq \Delta(G') + 1.
$$

If $G$ is connected, then the equality holds if and only if $\Delta(G') = n - 1$.

• **Lemma 2.5** [9] Let $G$ be a connected graph of order $n$ with a cutpoint $v$. Then

$$
\mu_{n-1}(G') \leq 1, \text{ with equality if and only if } v \text{ is adjacent to every vertex of } G.
$$
Proofs of Main results

• Lemma 2.6 Let $G$ be a connected graph of order $n \geq 3$ with two pendant vertices $u, v$ adjacent to a common vertex $w$. Then

$$\varphi_L(G + uv) \leq \varphi_L(G).$$

Proof. From the Corollary 3.9 of [13], we can get that $1$ is in $SL(G)$ and $SL(G + uv)$ is $SL(G) \setminus \{1\} \cup \{3\}$. Since the largest eigenvalue in $SL(G)$ is at least $\Delta(G) + 1 \geq 3$, the result follows.

• **Lemma 2.7** Let $G$ be any of the graphs $G_1(n-7; n)$, $n \geq 7$; $G_3(0, n - 6; n)$, $n \geq 6$; $G_4(0, n - 5; n)$, $n \geq 6$; $G_8(0, n - 5; n)$, $n \geq 6$; and $G_{18}(0, n - 4; n)$, $n \geq 5$. Then $\varphi_L(G) = n - 1$.

![Graphs G_1, G_3, G_4, G_8, G_{18}](image)

**Proof.** By Lemma 2.4 and Lemma 2.5, we can get the result easily.

• **Lemma 2.4[7]** Let $G$ be a graph of order $n \geq 2$ containing at least one edge. Then $\mu_1(G) \geq \Delta(G) + 1$. If $G$ is connected, then the equality holds if and only if $\Delta(G) = n - 1$.

• **Lemma 2.5[9]** Let $G$ be a connected graph of order $n$ with a cutpoint $v$. Then $\mu_{n-1}(G) \leq 1$, with equality if and only if $v$ is adjacent to every vertex of $G$. 
Lemma 2.8 Let $G$ be a connected tricyclic graph with a triangle attached at a single vertex. Then $\varphi_L(G) \leq n - 1$, the equality holds if and only if $G$ is $G_1(n - 7; n), n \geq 7$ or $G_3(0, n - 6; n), n \geq 6$.

We introduce nineteen tricyclic graphs of order $n$ in the following Figure 1.
Lemma 2.9 Let $G$ be one with maximum Laplacian spread of all tricyclic graphs of order $n \geq 11$. Then $G$ is among the graphs $G_1(n - 7; n)$, $G_2(1, n - 7; n)$, $G_3(0, n - 6; n)$, $G_3(1, n - 7; n)$, $G_4(0, n - 5; n)$, $G_4(1, n - 6; n)$, $G_5(0, n - 5; n)$, $G_6(1, n - 6; n)$, $G_7(1, n - 6; n)$, $G_8(0, n - 5; n)$, $G_8(1, n - 6; n)$, $G_9(0, n - 7; n)$, $G_{11}(0, n - 6; n)$, $G_{12}(0, n - 6; n)$, $G_{18}(0, n - 4; n)$, $G_{18}(1, n - 5; n)$, $G_{19}(n - 6, 1; n)$.

Proof. Let $v_iv_j$ be an edge of $G$. Then

$$d(v_i) + d(v_j) - |N(v_i) \cap N(v_j)| = |N(v_i) \cup N(v_j)| \leq n,$$

with equality holds if and only if $v_iv_j$ is adjacent to every vertex of $G$. Therefore, if $G$ has no edge that is adjacent to every vertex of $G$, then by Lemma 2.2, $\mu_1(G) \leq n - 1$ and hence $\varphi_L(G) = \mu_1(G) - \mu_{n-1}(G) < n - 1$ as $\mu_{n-1}(G) > 0$. In addition, if $G$ is a tricyclic graph with a triangle attached at a single vertex but not the graphs $G_1(n - 7; n)$ and $G_3(0, n - 6; n)$, then by Lemma 2.8, $\varphi_L(G) < n - 1$. However, by Lemma 2.7, $\varphi_L(G_1(n - 7; n)) = \varphi_L(G_3(0, n - 6; n)) = \varphi_L(G_4(0, n - 5; n)) = \varphi_L(G_8(0, n - 5; n)) = \varphi_L(G_{18}(0, n - 4; n)) = n - 1$. So
\(G\) must be one graph in Figure 1 for some \(r\) or \(s\).

For the graph \(G_2(r, s; n)\) of Figure 1 with \(1 \leq r \leq n - 6, 0 \leq s \leq n - 7\), by Lemma 2.3,

\textbf{Lemma 2.3} [12] Let \(G\) be a connected graph with vertex set \(\{v_1, v_2, ..., v_n\} (n \geq 2)\). Then \(\mu_1(G) \leq \max\{d(v_i) + m(v_i) : v_i \in V(G)\}\), where

\[m(v_i) = \frac{\sum_{v_j \in N(v_i)} d(v_j)}{d(v_i)},\]

the average of the degrees of the vertices adjacent to \(v_i\).)

\[\mu_1(G_2(r, s; n)) \leq \max\{r + 1 + \frac{n - 1}{r + 1}, s + 5 + \frac{n + 5}{s + 5}\}.\]
For \( n \geq 11 \), \( s \leq n - 8 \) and an arbitrary \( r \geq 1 \),

\[
  r + 1 + \frac{n - 1}{r + 1} \leq \max \{2 + \frac{n - 1}{2}, n - 5 + \frac{n - 1}{n - 5}\} \leq n - 1,
\]

\[
  s + 5 + \frac{n + 5}{s + 5} \leq \max \{5 + \frac{n + 5}{5}, n - 3 + \frac{n + 5}{n - 3}\} \leq n - 1,
\]

and hence \( \mu_1(G_2(r, s; n)) \leq n - 1 \), \( \varphi_L(G_2(r, s; n)) < n - 1 \) as \( \mu_{n-1}(G) > 0 \).

.....By the above discussion, if \( G \) is one with maximum Laplacian spread of all tricyclic graphs of order \( n \geq 11 \), then \( G \) is among the graphs \( G_1(n - 7; n) \), \( G_2(1, n - 7; n) \), \( G_3(0, n - 6; n) \), \( G_3(1, n - 7; n) \), \( G_4(0, n - 5; n) \), \( G_4(1, n - 6; n) \), \( G_5(0, n - 5; n) \), \( G_6(1, n - 6; n) \), \( G_7(1, n - 6; n) \), \( G_8(0, n - 5; n) \), \( G_8(1, n - 6; n) \), \( G_9(0, n - 7; n) \), \( G_{11}(0, n - 6; n) \), \( G_{12}(0, n - 6; n) \), \( G_{18}(0, n - 4; n) \), \( G_{18}(1, n - 5; n) \), \( G_{19}(n - 6, 1; n) \). The result follows. \( \square \)

We next show that except the graphs \( G_1(n - 7; n) \), \( G_3(0, n - 6; n) \), \( G_4(0, n - 5; n) \), \( G_8(0, n - 5; n) \) and \( G_{18}(0, n - 4; n) \), the Laplacian spreads of the other graphs
in Lemma 2.9 are all less than $n - 1$ for a suitable $n$. Thus by a little computation for the graphs in Figure 1 of small order, $G_1(n - 7; n), n \geq 7; G_3(0, n - 6; n), n \geq 6; G_4(0, n - 5; n), n \geq 6; G_8(0, n - 5; n), n \geq 6$; and $G_{18}(0, n - 4; n), n \geq 4$ are proved to be the only tricyclic graphs with maximum Laplacian spread among all tricyclic graphs of fixed order $n$.

**Lemma 2.10** For $n \geq 7, \varphi_L(G_2(1, n - 7; n)) < n - 1$.

**Proof.** The characteristic polynomial $det(\lambda I - L(G_2(1, n - 7; n)))$ of $L(G_2(1, n - 7; n))$ is

$$
\lambda(\lambda - 3)(\lambda^2 - 6\lambda + 7)(\lambda - 1)^{n-7}[\lambda^3 - (n + 2)\lambda^2 + (3n - 2)\lambda - n].
$$

![Diagram of graphs G2(r,s;n) and G2(1,n-7;n)]
By Lemma 2.1 and Lemma 2.4, \( n > \mu_1 > n - 1 \geq 6 \), and by Lemma 2.5, \( \mu_{n-1} < 1 \). So \( \mu_1, \mu_{n-1} \) are both roots of the following polynomial: \( f_1(\lambda) = \lambda^3 - (n + 2)\lambda^2 + (3n - 2)\lambda - n \).

Observe that
\[
(n - 1) - \varphi_L(G_2(1, n - 7; n)) = (n - 1) - (\mu_1 - \mu_{n-1}) = (n - \mu_1) - (1 - \mu_{n-1}).
\]

If we can show \( n - \mu_1 > 1 - \mu_{n-1} \), the result will follow. By Lagrange Mean Value Theorem, \( f_1(n) - f_1(\mu_1) = (n - \mu_1)f_1'(\xi_1) \) for some \( \xi_1 \in (\mu_1, n) \). As \( f_1'(x) \) is positive and strict increasing on the interval \( (\mu_1, n) \],

\[
n - \mu_1 = \frac{f_1(n) - f_1(\mu_1)}{f_1'(\xi_1)} > \frac{n^2 - 3n}{f_1'(n)} = 1 - \frac{2n - 2}{n^2 - n - 2},
\]

Note that the function \( g_1(x) = \frac{2x - 2}{x^2 - x - 2} \) is strictly decreasing for \( x \geq 7 \). Hence

\[
(n - \mu_1) - (1 - \mu_{n-1}) > \mu_{n-1} - g_1(n) \geq \mu_{n-1} - g_1(7) = \mu_{n-1} - 0.3.
\]
Observe that a star of order $n$ has eigenvalues: $0$, $n$, $1$ of multiplicity $n - 2$, and hence has $n - 1$ eigenvalues not less than $1$. As $G_2(1, n - 7; n)$ contains a star of order $n - 1$, by eigenvalues interlacing theorem (that is, $\mu_i(G) \geq \mu_i(G - e)$ for $i = 1, 2, \ldots, n$ if we delete an edge $e$ from a graph $G$ of order $n$; or see [14]), $G_2(1, n - 7; n)$ has $(n - 2)$ eigenvalues not less than $1$. Now $f_1(0.3) = -0.753 - 0.19n < 0$ and $f_1(1) = n - 3 > 0$. So $0.3 < \mu_{n-1} < 1$. The result follows. □

**Lemma 2.11** For $n \geq 7$ $\varphi_L(G_3(1, n - 7; n)) < n - 1$.

**Lemma 2.12** For $n \geq 9$, $\varphi_L(G_4(1, n - 6; n)) < n - 1$.

**Lemma 2.13** For $n \geq 7$, $\varphi_L(G_5(0, n - 5; n)) < n - 1$.

**Lemma 2.14** For $n \geq 8$, $\varphi_L(G_6(1, n - 6; n)) < n - 1$.

**Lemma 2.15** For $n \geq 8$, $\varphi_L(G_7(1, n - 6; n)) < n - 1$.

**Lemma 2.16** For $n \geq 8$, $\varphi_L(G_8(1, n - 6; n)) < n - 1$. 
Lemma 2.17 For $n \geq 7$, $\varphi_L(G_9(0, n-7; n)) < n - 1$.

Lemma 2.18 For $n \geq 8$, $\varphi_L(G_{11}(0, n-6; n)) < n - 1$.

Lemma 2.19 For $n \geq 8$, $\varphi_L(G_{12}(0, n-6; n)) < n - 1$.

Lemma 2.20 For $n \geq 7$, $\varphi_L(G_{18}(1, n-5; n)) < n - 1$.

Lemma 2.21 For $n \geq 7$, $\varphi_L(G_{19}(n-6, 1; n)) < n - 1$.

From the previous discussion, we can get that $G_1(n-7; n)$, $G_3(0, n-6; n)$, $G_4(0, n-5; n)$, $G_8(0, n-5; n)$ and $G_{18}(0, n-4; n)$ of Figure 1 are the only five graphs with maximum Laplacian spread among all tricyclic graphs of order $n \geq 11$.

Moreover, for $5 \leq n \leq 10$, if $G$ is one with maximum Laplacian spread of all tricyclic graphs of order $n$, then $G$ is necessary among the graphs in Figure 1 (by the first paragraph of the proof of Lemma 2.9), and we can determine that the Laplacian spreads of the graphs in Figure 1 are all less than $n - 1$ (by Lemma 2.3 and Lemmas 2.10-2.21) except for the graphs shown in Figure 3.
By a little computation (with Mathematica) for the graphs in Figure 3, we find that $G_1(n - 7; n)$, $7 \leq n \leq 10$; $G_3(0, n - 6; n)$, $6 \leq n \leq 10$; $G_4(0, n - 5; n)$, $6 \leq n \leq 10$; $G_8(0, n - 5; n)$, $6 \leq n \leq 10$; and $G_{18}(0, n - 4; n)$, $5 \leq n \leq 10$ of Figure 1 are the only graphs with maximum Laplacian spread among all tricyclic graphs of order $n$ for $5 \leq n \leq 10$.

\begin{center}
\begin{tabular}{|l|l|l|l|l|}
\hline
$n$ & $G_4(0, 0; 5)$ & $G_7(0, 0; 5)$ & $G_{18}(0, 1; 5)$ \\
\hline
5 & $2 + \sqrt{2}$ & 3 & 4 \\
\hline
$n$ & $G_3(0, 0; 6)$ & $G_4(0, 1; 6)$ & $G_4(1, 0; 6)$ & $G_5(0, 1; 6)$ & $G_8(0, 1; 6)$ & $G_8(1, 0; 6)$ \\
\hline
6 & 5 & 5 & 4.3871 & 4.4177 & 5 & $2\sqrt{5}$ \\
\hline
$n$ & $G_{10}(0, 0; 6)$ & $G_{11}(0, 0; 6)$ & $G_{12}(0, 0; 6)$ & $G_{18}(0, 2; 6)$ & $G_{18}(1, 1; 6)$ & $G_{19}(0, 1; 6)$ \\
\hline
7 & 4 & $\sqrt{3} + \sqrt{2} + 1$ & 4.1696 & 5 & $2\sqrt{5}$ & 4.6002 \\
\hline
$n$ & $G_1(0; 7)$ & $G_3(0, 1; 7)$ & $G_4(0, 2; 7)$ & $G_4(1, 1; 7)$ & $G_4(2, 0; 7)$ & $G_5(1, 1; 7)$ & $G_6(1, 1; 7)$ \\
\hline
7 & 6 & 6 & 6 & 5.3905 & 4.8682 & 4.7921 & 5.3141 \\
\hline
\end{tabular}
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<td>(G_9(1, 1; 9))</td>
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<th>(G_{12}(1, 2; 9))</th>
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<th>(G_{18}(0, 5; 9))</th>
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\(n=10\)

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<th>(G_1(3; 10))</th>
<th>(G_2(2, 2; 10))</th>
<th>(G_3(0, 4; 10))</th>
<th>(G_3(2, 2; 10))</th>
<th>(G_4(0, 5; 10))</th>
<th>(G_4(2, 3; 10))</th>
<th>(G_6(2, 3; 10))</th>
</tr>
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<td>spread</td>
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<td>7.7142</td>
<td>9</td>
<td>7.6058</td>
<td>9</td>
<td>7.5591</td>
<td>7.4849</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>graph</th>
<th>(G_7(2, 3; 10))</th>
<th>(G_8(0, 5; 10))</th>
<th>(G_8(2, 3; 10))</th>
<th>(G_{18}(0, 6; 10))</th>
<th>(G_{18}(2, 4; 10))</th>
<th>(G_{19}(3, 2; 10))</th>
</tr>
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<tbody>
<tr>
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<td>9</td>
<td>7.5437</td>
<td>9</td>
<td>7.5212</td>
<td>6.7302</td>
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</tbody>
</table>

Figure 3 Laplacian spreads of some graphs of order \(n\) in Figure 1 for \(5 \leq n \leq 10\).
Main result

Theorem 2.22 \(G_1(n - 7; n), n \geq 7; G_3(0, n - 6; n), n \geq 6; G_4(0, n - 5; n), n \geq 6; G_8(0, n - 5; n), n \geq 6; \) and \(G_{18}(0, n - 4; n), n \geq 4\) of Figure 1 (the following figure) are the only graphs with maximum Laplacian spread among all tricyclic graphs of fixed order \(n\). For each \(n \geq 5\), the maximum Laplacian spread is equal to \(n - 1\).

Remark There is only one tricyclic graph of order \(n \leq 4\). It is \(G_{18}(0, 0; 4) = K_4\) with Laplacian spread 0.

\[
\begin{align*}
G_1(n - 7; n) & \quad G_3(0, n - 6; n) & \quad G_4(0, n - 5; n) & \quad G_8(0, n - 5; n) & \quad G_{18}(0, n - 4; n) \\
n-7 & \quad n-6 & \quad n-5 & \quad n-5 & \quad n-4
\end{align*}
\]
Thank you.